Bilkent University
Department of Mathematics

Problem Of The Month

October 2008

Problem:

Let $\mathbb{Z}$ be the set of all integers. Prove that there is no function $f : \mathbb{Z} \to \mathbb{Z}$ such that for any $m, n \in \mathbb{Z}$

$$f(n) - f(n + f(m)) = m \qquad (1)$$

Solution:

f (if exists) takes different values at different points: $f(n + f(m_1)) = f(n) - m_1$ and $f(n + f(m_2)) = f(n) - m_2$. Therefore, $f(m_1) = f(m_2)$ implies $m_1 = m_2$.

f takes all integer values: put $m = 0$ in (1): $f(n + f(0)) = f(n)$. Thus, $f(0) = 0$. Again put $n = 0$ in (1): $f(f(m)) = -m$.

Since $f(n+m) = f(n+f(f(-m))) = f(n) - f(f(-m)) = f(n) + f((f(f-m))) = f(n) + f(m)$ we have $f(n) = cn$. Put $f(n) = cn$ in $f(f(m)) = -m$: $c^2 n = -n$ or $c^2 = -1$. Done.