Bilkent University
Department of Mathematics

Problem Of The Month

June 2008

Problem:

Some unit squares of 2008×2008 square board are colored. Let (i, j) be a unit square belonging to the i^{th} line and j^{th} column and $S_{i,j}$ be the set of all colored unit squares (x, y) satisfying $x \leq i$ and $y \leq j$. At the first step in each colored unit square (i, j) we write the number of colored unit squares in $S_{i,j}$. In each step, in each colored unit square (i, j) we write the sum of all numbers written in $S_{i,j}$ in the previous step. Prove that after finite number of steps, all numbers in the colored unit squares will be odd.

Solution:
Let $f_{(i,j)}$ be the number written on (i, j) in $\mod 2$. We can suppose that at the 0^{th} step $f_{(i,j)} = 1$ for all colored unit squares (i, j). We prove the statement by induction with respect to n, the total number of colored unit squares. If $n = 1$, then after the first step $f_{(i,j)} = 1$ for the only colored unit square (i, j). Suppose that the statement is proved for $n = k$ and consider the case $n = k + 1$. Let us define a partial order between colored unit squares: we say that $(i, j) \preceq (k, l)$ if $i \leq k$ and $j \leq k$. Let (p, q) be any maximal element with respect to this order (there is at least one maximal element). The unit square (p, q) has no any influence on other colored unit squares at any step. If we remove (p, q), then by inductive hypothesis, after N steps on each of remaining unit squares the number 1 will be written. Therefore, if we do not remove (p, q), after N steps $f_{(i,j)} = 1$ for all unit squares except possibly for $f_{(p,q)}$. If $f_{(p,q)} = 1$, it is done. Suppose that $f_{(p,q)} = 0$. It means that the first N steps have added 1 to $f_{(p,q)}$ and it is changed from 1 to 0. Therefore, after $2N$ steps $f_{(i,j)} = 1$ for all colored unit squares.