Bilkent University
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Problem Of The Month

February 2008

Problem:
Suppose that the polynomial $P(x) = x^{2008} + a_{2007}x^{2007} + a_{2006}x^{2006} + \cdots + a_1 x + a_0$ has 2008 real roots, while the polynomial $P(Q(x))$, where $Q(x) = \dfrac{x^2}{4} + x - 1$ has no real root. Prove that $a_0 + a_1 + \cdots + a_{2007} > 3^{2008} - 1$.

Solution:

Suppose that roots of the polynomial $P(x)$ are $x_1, x_2, \ldots, x_{2008}$. Then $P(x) = \prod_{i=1}^{2008}(x - x_i)$ and $P(Q(x)) = \prod_{i=1}^{2008}(Q(x) - x_i)$. Therefore, for any x, $Q(x) = \dfrac{x^2}{4} + x - 1 \neq x_i$ or $x_i < -2$. Now $a_0 + a_1 + \cdots + a_{2007} = P(1) - 1 = \prod_{i=1}^{2008}(1 - x_i) - 1 \geq 3^{2008} - 1$. Done.