Bilkent University
Department of Mathematics

Problem Of The Month

January 2008

Problem:
Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that $f(2007) = 2008$ and for each $x, y \in \mathbf{R}$

$$f(4xy) = 2y(f(x+y) + f(x-y)).$$

Solution:

Let us prove that the only solution is $f(x) = \frac{2008}{2007}x$.

$y = 0$ in our equation gives $f(0) = 0$ and $x = 0$ yields $0 = f(y) + f(-y)$, so the function f is odd. Since $f(4st) = f(4ts)$ we get $2t(f(s+t) + f(s-t)) = 2s(f(s+t) - f(s-t))$ or

$$(s-t)f(s+t) = (s+t)f(s-t).$$

We take $t = 2007 - s$ and use $f(2007) = 2008$:

$$(2s-2007)2008 = 2007f(2s-2007).$$

Substitution $2s - 2007 = x$ gives $f(x) = \frac{2008}{2007}x$. Finally, we verify that the function $f(x) = \frac{2008}{2007}x$ satisfies the equation.