Bilkent University
Department of Mathematics

Problem Of The Month

November 2007

Problem:
Let a, b, c be non-negative real numbers satisfying $a + b + c = 5$. Find the maximal value of $a^4b + b^4c + c^4a$.

Solution:

The maximal value is 256 and is attained at $(4, 1, 0)$, $(0, 4, 1)$ or $(1, 0, 4)$.

Define $f(x, y, z) = x^4y + y^4z + z^4x$. Let $a \geq b$ and $a \geq c$. Let us prove that $f(a + c/2, b + c/2, 0) \geq f(a, b, c)$. Indeed,

$$f(a+c/2, b+c/2, 0) = (a+c/2)^4(b+c/2) \geq (a^4+2a^3c)(b+c/2) \geq a^4b+2a^3bc+a^3c^2$$
$$\geq a^4b + b^4c + c^4a = f(a, b, c)$$

Now we maximize $f(a, b, o)$ when $a + b = 5$ by using of AM-GM inequality:

$$5 = a + b = (a/4 + a/4 + a/4 + a/4 + b) \geq 5 \times \sqrt[5]{a^4b \cdot 4^4}$$

Therefore, $a^4b \leq 4^4$. Equality holds at $a = 4, b = 1$. Similarly we obtain other maximum triples $(0, 4, 1)$ and $(1, 0, 4)$ when maximum of a, b, c is b and c. Done.