Bilkent University
Department of Mathematics

Problem Of The Month

September 2007

Problem: Find the smallest real number A such that for all triangle angles α, β and γ the inequality

$$\sin^2 \alpha + \sin^2 \beta - \cos \gamma \leq A$$

holds.

Solution:
The answer is $\frac{5}{4}$.

Let us prove that

$$f = \sin^2 \alpha + \sin^2 \beta - \cos \gamma < \frac{5}{4} \tag{1}$$

Indeed, $f = 2 - \cos^2 \alpha - \cos^2 \beta - \cos \gamma = 1 - \frac{1}{2}(\cos 2\alpha + \cos 2\beta) + \cos(\alpha + \beta)$ $= 1 - (\cos(\alpha + \beta)(\cos(\alpha - \beta) - 1)$
and the inequality (1) is equivalent to

$$(\cos(\alpha + \beta)(1 - \cos(\alpha - \beta)) < \frac{1}{4} \tag{2}$$

(2) follows from the inequality $ab \leq \frac{(a+b)^2}{4}$. Indeed,
$(\cos(\alpha+\beta)(1-\cos(\alpha-\beta)) \leq \frac{1}{4}(\cos(\alpha+\beta)+1-\cos(\alpha-\beta))^2 = \frac{1}{4}(1-2\sin\alpha\sin\beta)^2 < \frac{1}{4}$, since $0 < \sin\alpha\sin\beta < 1$ (α, β are triangle angles).

(1) is proved. Now note that if $\gamma = \dfrac{2\pi}{3}$ and α approaches to $\dfrac{\pi}{3}$, then f approaches to $\frac{5}{4}$. Done.