Bilkent University
Department of Mathematics

Problem Of The Month

July-August 2007

Problem: For all positive real numbers a, b, c satisfying $a + b + c = 1$, prove the following inequality:

$$\frac{1}{ab + 2c^2 + 2c} + \frac{1}{bc + 2a^2 + 2a} + \frac{1}{ca + 2b^2 + 2b} \geq \frac{1}{ab + bc + ca}$$

Solution:

Note that

$$\frac{1}{ab + 2c^2 + 2c} \geq \frac{ab}{(ab + bc + ca)^2} \tag{1}$$

Indeed, since $a^2 + b^2 \geq 2ab$ and $a + b + c = 1$,

$(ab + bc + ca)^2 = a^2b^2 + b^2c^2 + c^2a^2 + 2a^2bc + 2b^2ca + 2c^2ab = a^2b^2 + (a^2 + b^2)c^2 + 2abc(a + b + c) \geq a^2b^2 + 2abc^2 + 2abc = ab(ab + 2c^2 + 2c)$.

We obtain cyclicly two more inequalities from (1). The sum of these three inequalities gives the required inequality.