Bilkent University
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Problem Of The Month

April 2007

Problem: Find all positive odd integers n for which there exist odd integers $x_1, x_2, \ldots, x_n$ such that

$$x_1^2 + x_2^2 + \cdots + x_n^2 = n^4.$$

Solution: The answer is $n = 8k+1$, k is a nonnegative integer number.

If $n = 1$, $x_1 = 1$ satisfies the conditions. If $n > 1$, then for any odd x, $x^2 \equiv 1(\mod 8)$ and therefore,

$$x_1^2 + x_2^2 + \cdots + x_n^2 \equiv 1 + 1 + \cdots + 1 = n$$

and $n^4 = (n^2)^2 \equiv 1(\mod 8)$. Thus, $n \equiv 1(\mod 8)$. Let us show that if $n = 8k+1$, then there are n numbers $x_1, x_2, \ldots, x_n$ satisfying the condition. Indeed,

$$(8k+1)^4 = (8k-1)^4 + (8k+1)^4 - (8k-1)^4$$
$$= (8k-1)^4 + ((8k+1)^2 - (8k-1)^2)((8k+1)^2 + (8k-1)^2)$$
$$(8k-1)^4 + 32k(128k^2 + 2)$$
$$(8k-1)^4 + 4k(32k-1)^2 + (16k-1)^2 + (92k-1)$$
$$= (8k-1)^4 + 4k(32k-1)^2 + (16k-1)^2 + 92(k-1) + 91$$
$$= (8k-1)^4 + 4k(32k-1)^2 + (16k-1)^2 + (k-1)(9^2 + 3^2 + 1^2 + 1^2) + (9^2 + 3^2 + 1^2)$$

and in the last line there are $1 + 4k + 1 + 4(k-1) + 3 = 8k+1$ odd squares. Done. Of course, there are many other ways for the representation of $(8k+1)^4$ as a sum of $8k+1$ squares. For example (due to Tomas Jurik):

$$(8k+1)^4 = (64k^2 + 16k + 1)^2 = ((64k^2 + 16k - 1 + 2)^2$$

$$= (64k^2+16k-1)^2 + 4(64k^2+16k-1) + 4 = (64k^2+16k-1)^2 + 16^2k^2 + 64k$$
$$= (64k^2+16k-1)^2 + (16k+1)^2 + 32k - 1$$
$$= (64k^2+16k-1)^2 + (16k+1)^2 + k(5^2+1^2+1^2+1^2+1^2+1^2+1^2) + \underbrace{1^2+\cdots+1^2}_{(k-1) times}$$

and in the last line there are $2 + 7k + (k-1) = 8k + 1$ odd squares.