Bilkent University
Department of Mathematics

Problem Of The Month

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Problem: Suppose that $A_1, A_2, \ldots, A_n$ are points on the plane located in such a way that for any point P on the same plane at least one of the distances $dist(P, A_i)$, $i = 1, 2, \ldots, n$ is irrational. Find the minimal possible value of n.

Solution: The answer is 3.

Consider a line L perpendicular to the line segment $[A_1A_2]$ and passing through the center of the segment $[A_1A_2]$. It is clear that there are infinitely many points P located on L such that the distance between P and A_1 (so between P and A_2) is rational. Therefore, $n \geq 3$. Let us show that $n = 3$.
Let A_1 and A_2 be two points on the plane with $dist(A_1, A_2) = \sqrt[4]{2}$ and A_3 be the center of $[A_1A_2]$. Let us show that for any point P on the same plane at least one of the distances $dist(P, A_i)$, $i = 1, 2, 3$ is irrational. If P lies on the line passing through A_1 and A_2, then obviously one of these distances is irrational. Suppose that P does not belong to this line. Consider the parallelogram with vertices A_1, P, A_2 and Q (Q is uniquely determined by A_1, P, A_2). Then

$$|A_1A_2|^2 + |PQ|^2 = 2|PA_1|^2 + 2|PA_2|^2,$$

or

$$|A_1A_2|^2 = 2|PA_1|^2 + 2|PA_2|^2 - 4|PA_3|^2. \qquad (*)$$

Since $|A_1A_2|^2 = (\sqrt[4]{2})^2 = \sqrt{2}$ is irrational, at least one of the terms in $(*)$ is irrational. Therefore, at least one of the distances $|PA_i|$, $i = 1, 2, 3$ is irrational. Done.