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Problem Of The Month

February 2007

Problem: Let P be the product of the positive real numbers $a_1, a_2, \ldots, a_{1024}$. Prove that

$$\prod_{i=1}^{1024} (1 + \frac{1}{a_i^{1024} + a_i^{2048}}) \geq (1 + \frac{1}{P + P^2})^{1024}$$

Solution:

Let us prove that for arbitrary positive real numbers a, b the following inequality holds:

$$\left(1 + \frac{1}{a^2 + a^4}\right)\left(1 + \frac{1}{b^2 + b^4}\right) \geq \left(1 + \frac{1}{ab + a^2b^2}\right)^2 \qquad (*)$$

Let us multiply both sides of $(*)$ by $a^2b^2(1+a^2)(1+b^2)(1+ab)^2$:

$$(1 + a^2 + a^4)(1 + b^2 + b^4)(1 + ab)^2 \geq (1 + a^2)(1 + b^2)(1 + ab + a^2b^2)^2.$$

After expansion of the brackets (27+24 terms!) and simplifying the inequality $(*)$ has the following form:

$$a^4 + b^4 - 2a^2b^2 + 2ab^5 + 2a^5b - a^2b^4 - a^4b^2 - 2a^3b^3 + a^2b^6 + a^6b^2 - 2a^4b^4 \geq 0$$

which is equivalent to

$$(a^2 - b^2)^2 + ab(2b^4 + 2a^4 - ab^3 - a^3b - 2a^2b^2) + a^2b^2(a^2 - b^2)^2$$

$$= (a^2-b^2)^2 + ab((a^2-b^2)^2 + (a-b)^2(a^2+ab+b^2)) + a^2b^2(a^2-b^2)^2 \geq 0$$

Since all terms of the last inequality are nonnegative, the inequality $(*)$ is proved.

Now, we prove by induction that for all $n \geq 1$ and any positive real numbers $a_1, a_2, \ldots, a_{2^n}$ with product P_n

$$\prod_{i=1}^{2^n}\left(1+\frac{1}{a_i^{2^n}+a_i^{2^{n+1}}}\right) \geq \left(1+\frac{1}{P_n+P_n^2}\right)^{2^n} \qquad (I_n)$$

In particular, this will give the required inequality when $n = 10$.
Inequality (I_1) directly follows from $(*)$. Assume that (I_n) holds. Then

$$\prod_{i=1}^{2\times 2^n}\left(1+\frac{1}{a_i^{2\times 2^n}+a_i^{4\times 2^n}}\right) = \prod_{i=1}^{2^n}\left(1+\frac{1}{a_{2i-1}^{2\times 2^n}+a_{2i-1}^{4\times 2^n}}\right)\left(1+\frac{1}{a_{2i}^{2\times 2^n}+a_{2i}^{4\times 2^n}}\right)$$

$$\geq \prod_{i=1}^{2^n}\left(1+\frac{1}{a_{2i-1}^{2^n}a_{2i}^{2^n}+a_{2i-1}^{2\times 2^n}a_{2i}^{2\times 2^n}}\right)^2 (by(*)) \geq \left[\left(1+\frac{1}{\prod_{i=1}^{2^n} a_{2i-1}a_{2i}+(\prod_{i=1}^{2^n} a_{2i-1}a_{2i})^2}\right)^{2^n}\right]^2$$

(the latter from (I_n) applied to the numbers $a_{2i-1}a_{2i}$ $(i = 1, 2, \ldots, 2^n)$). Thus,

$$\prod_{i=1}^{2\times 2^n}\left(1+\frac{1}{a_i^{2\times 2^n}+a_i^{4\times 2^n}}\right) \geq \left(1+\frac{1}{\prod_{i=1}^{2\times 2^n} a_i+(\prod_{i=1}^{2\times 2^n} a_i)^2}\right)^{2\times 2^n}$$

The last inequality is (I_{n+1}). The proof is completed.

The problem has an alternative solution based on the convexity of the function $\ln(1+\frac{1}{\exp x+\exp 2x})$.