Bilkent University
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Problem Of The Month

January 2007

Problem: Find the greatest common divisor of natural numbers a and b satisfying

$$(1+\sqrt{2})^{2007} = a + b\sqrt{2}.$$

Solution: The answer is 1.

Note that $(1+\sqrt{2})^{2007}(1-\sqrt{2})^{2007} = -1$.
If we expand $(1+\sqrt{2})^{2007}$ and $(1-\sqrt{2})^{2007}$ by Newton's binomial formula we readily see that $(1-\sqrt{2})^{2007} = a - b\sqrt{2}$ since irrational terms of the expansions are terms with odd power of $\sqrt{2}$.
Therefore, $(1+\sqrt{2})^{2007}(1-\sqrt{2})^{2007} = (a+b\sqrt{2})(a-b\sqrt{2}) = a^2 - 2b^2$. Thus, $a^2 - 2b^2 = -1$. Now, if d is a greatest common divisor of a and b, then $d|a^2$ and $d|b^2$ and therefore $d|(a^2 - 2b^2) = -1$. Thus, $d = 1$.

The problem also has a simple solution by the method of mathematical induction:

Solution 2:

For $n = 1, 2, \ldots$, define the natural numbers a_n and b_n by $(1+\sqrt{2})^n = a_n + b_n\sqrt{2}$.
We prove that the greatest common divisor of a_n and b_n is equal to 1 for all $n = 1, 2, \ldots$.
Clearly, $a_1 = b_1 = 1$. From

$$\begin{aligned} a_{n+1} + b_{n+1}\sqrt{2} = (1+\sqrt{2})^{n+1} &= (1+\sqrt{2})^n(1+\sqrt{2}) = (a_n + b_n\sqrt{2})(1+\sqrt{2}) \\ &= a_n + 2b_n + (a_n + b_n)\sqrt{2} \end{aligned}$$

it follows that

$$a_{n+1} = a_n + 2b_n, \qquad b_{n+1} = a_n + b_n$$

Now, any common divisor of a_{n+1} and b_{n+1} also divides $2b_{n+1} - a_{n+1} = a_n$ and $a_{n+1} - b_{n+1} = b_n$ and so is a common divisor of a_n and b_n. Therefore, $\gcd(a_{n+1}, b_{n+1}) = 1$ whenever $\gcd(a_n, b_n) = 1$. Since $\gcd(a_1, b_1) = 1$, it follows by induction that $\gcd(a_n, b_n) = 1$ for all positive integers n. Particularly, $\gcd(a, b) = \gcd(a_{2007}, b_{2007}) = 1$.