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Problem Of The Month

December 2006

Problem: Let $\{a_n\}$ be a sequence of natural numbers such that for each $n \geq 1$ $a_{n+1} = \sum_{k=1}^{n} a_k^2$ and a_{2006} is divisible by 2006. Find the smallest possible value of a_1.

Solution: The answer is 472.

Since $2006 = 2 \times 17 \times 59$, a_{2006} must be divisible by $2, 17$ and 59.

We will use the following auxiliary

Proposition. The equation $a^2 + a + 1 = 0 \mod (p)$ has no integer solutions for $p = 17$ and $p = 59$.

Proof can be obtained by *direct inspection* or for example by using of Legendre symbols:

for $p = 17$: The equation is equivalent to $a^2 + a - 17a + 1 + 63 = 63 \mod (17)$ or $(a-8)^2 = 12 \mod (17)$ which is impossible since $(\frac{12}{17}) = (\frac{3}{17})(\frac{4}{17}) = (\frac{3}{17}) = (\frac{17}{3}) = (\frac{2}{3}) = -1$.

for $p = 59$: The equation is equivalent to $a^2 + a + 59a + 1 + 899 = 899 \mod (59)$ or $(a-30)^2 = 14 \mod (59)$ which is impossible since $(\frac{14}{59}) = (\frac{2}{59})(\frac{7}{59}) = (\frac{59}{7}) = (\frac{3}{7}) = -1$. Done.

1 (divisibility by 2). For all $k \geq 3$ $a_k = a_{k-1}^2 + a_{k-1} = a_{k-1}(a_{k-1}+1)$. Therefore, for all values of a_1 a_{2006} is even.

2 (divisibility by 59). Let m be a minimal natural number such that a_m is divisible by 59 (since a_{2006} is divisible by 59, m is well defined). Suppose that $m \geq 4$. Then $59|a_m$ and $59 \not| a_{m-1}$. Since $a_m = a_{m-1}(a_{m-1}+1)$ we readily have $59|(a_{m-1}+1)$ or $a_{m-1} = -1 \mod (59)$. Since $a_{m-1} = a_{m-2}^2 + a_{m-2}$ and $a_{m-1} = -1 \mod (59)$ we get an equation $a_{m-2}^2 + a_{m-2} + 1 = 0 \mod (59)$. Impossible by the Proposition. Therefore, $m \leq 3$. But $m \neq 3$, since as above $a_2 = -1 \mod (59)$ and since $a_2 = a_1^2$ we have $a_1^2 = -1 \mod (59)$, impossible since $59 = 4 \times 14 + 3$. Finally, $m \neq 2$, since $59|a_1^2$ implies that $59|a_1$. Thus, a_1 is divisible by 59.

3 (divisibility by 17). Let m be a minimal natural number such that a_m is divisible by 17 (since a_{2006} is divisible by 17, m is well defined). Suppose that $m \geq 4$. Then $17|a_m$ and $17 \not| a_{m-1}$. Since $a_m = a_{m-1}(a_{m-1}+1)$ we readily have $17|(a_{m-1}+1)$ or $a_{m-1} = -1 \mod (17)$. Since $a_{m-1} = a_{m-2}^2 + a_{m-2}$ and $a_{m-1} = -1 \mod (17)$ we get an equation $a_{m-2}^2 + a_{m-2} + 1 = 0 \mod (59)$. Impossible by the Proposition. Therefore, $m \leq 3$. $m \neq 2$, since $17|a_1^2$ implies that $17|a_1$. If $m = 1$, then $a_1 \geq 17 \times 59 = 1003$. If $m = 3$, then $a_3 = a_1^2 + a_2^2$ and $a_2 = -1 \mod (17)$ imply that $a_1 = \pm 4 \mod (17)$.

As a result, $a_1 = 59l$. Therefore, $a_1 = 8l = \pm 4 \mod (17)$. The minimal natural solution of the last equation is $l = 8$. Thus, $a_1 \geq 59 \times 8 = 472$. Now we note that $a_1 = 472$ satisfies the condition, since starting $n \geq 3$ all terms of the sequence are divisible by 2006.