Bilkent University
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Problem Of The Month

November 2006

Problem: Let $x_1, x_2, \ldots, x_{2006}$ be positive real numbers satisfying the condition

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_{2006}} = 1.$$

Find the minimal possible value of the expression $x_1 x_2 \ldots x_{2006}$.

Solution: The answer is 2005^{2006}.

Let $t_i = \frac{1}{1+x_i}, i = 1, 2, \ldots, 2006$. Then $x_i = \frac{1-t_i}{t_i}$. In new variables we have to minimize the expression $\prod_{i=1}^{2006}(\frac{1-t_i}{t_i})$ given $\sum_{i=1}^{2006} t_i = 1$. Note that for each i, $i = 1, 2, \ldots, 2006$

$$\frac{1-t_i}{t_i} = \frac{t_1 + t_2 + \cdots + t_{i-1} + t_{i+1} + \cdots + t_{2006}}{t_i} \geq \frac{2005 \sqrt[2005]{t_1 t_2 \ldots t_{i-1} t_{i+1} \ldots t_{2006}}}{t_i}$$

by AG inequality. By multiplying all this inequalities we get

$$\prod_{i=1}^{2006}(\frac{1-t_i}{t_i}) \geq 2005^{2006} \frac{\sqrt[2005]{t_1^{2005} t_2^{2005} \ldots t_{2006}^{2005}}}{t_1 t_2 \ldots t_{2006}} = 2005^{2006}.$$

Now we note that if $t_i = \frac{1}{2006}$ for each $i = 1, 2, \ldots, 2006$, then $\prod_{i=1}^{2006}(\frac{1-t_i}{t_i}) = 2005^{2006}$.

Done.