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Problem Of The Month

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Problem: Let $a_1, a_2, \ldots, a_N$ be pairwise different positive integer numbers satisfying the following two conditions:

1. $a_i < 22$ for all $1 \le i \le N$.

2. $a_k + a_l \ne a_m + a_n$ for all pairwise different k, l, m, n.

Find the maximal possible value of N.

Solution: The answer is 7. It can be readily seen that the numbers $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 5, a_5 = 8, a_6 = 13$ and $a_7 = 21$ satisfy the conditions. Therefore, $N \ge 7$. Let us show that $N < 8$.

Let $a_1, a_2, \ldots, a_n$ be a collection satisfying the conditions. Consider all possible pairs (a_i, a_j), $a_i > a_j$. The difference $a_i - a_j$ takes values between 1 and 20.

Consider two pairs : (a_k, a_l) and (a_p, a_q). Suppose that $a_k - a_l = a_p - a_q$. Then $l = p$ (otherwise $a_k + a_q = a_p + a_l$ and the conditions are violated). In this case the number a_l will be called "common" for pairs (a_k, a_l) and (a_l, a_q). Now we note that a number a_l can be "common" just for these pairs (a_k, a_l) and (a_l, a_q). Indeed, if there are another pairs (a_r, a_l) and (a_l, a_s) then $a_r + a_s = a_k + a_q$ and again the conditions are violated. Also, minimal and maximal elements of the collection $a_1, a_2, \ldots, a_n$ can not be "common" numbers. Therefore, all possible pairs $a_i - a_j$ give at most $20 - 1 + n - 2 = 17 + n$ differences. The number of pairs is $\frac{n(n-1)}{2}$. Therefore, $17 + n \ge \frac{n(n-1)}{2}$ or $n^2 - 3n \le 36$, which implies that $n < 8$. Done.