Bilkent University
Department of Mathematics

Problem Of The Month

January 2006

Problem: Find all natural numbers m, n, and k satisfying the following equation:

$$5^m + 7^n = k^3.$$

Solution: Let (m, n, k) be a solution of the equation $5^m + 7^n = k^3$.

1. Let us prove that n is an odd number:
Indeed, k must be even, and therefore the right hand side is divisible by 8. Since $5^m = 1$ or 5 for even and odd values of m, and $7^n = 1$ or 7 for even and odd values of $n(mod 8)$, the only possibility is: m is even, n is odd.

2. Let us prove that m is divisible by 3:
Indeed, k is not divisible by 7, otherwise 7 divides 5^m. Therefore, $k^3 = 1$ or -1 $(mod 7)$. Thus, $5^m = 1$ or -1 $(mod 7)$. This is possible only for $m = 3l$.

3. Now we have

$$7^n = k^3 - 5^{3l} = (k - 5^l)(k^2 + 5^l k + 5^{2l}).$$

The second factor exceeds 3 and therefore is divisible by 7.

If the first factor $k - 5^{m'}$ is equal to 1, then $5^m + 7^n = k^3 = 1(mod 5)$ and since n is odd, $7^n = 1(mod 5)$, no solution for odd n.

If the first factor $k - 5^l$ is divisible by 7, then 7 also divides its square
$k^2 - 2 \cdot 5^{m'} k + 5^{2l}$ and since 7 also divides the second factor $k^2 + 5^l k + 5^{2l}$, 7 divides

their difference $3 \cdot 5^{m'} k$. Finally, since $5^m \neq 0 (mod 7)$, 7 must divide k. Again, since $5^m \neq 0 (mod 7)$ the equation $5^m + 7^n = k^3$ has no solution.
Thus, our equation has no solution in natural numbers (the only nonnegative integer solution is (0,1,2)).