Bilkent University
Department of Mathematics

Problem Of The Month

November 2005

Problem: Find all triples of natural numbers a, b, and c, such that

$$ab + c = (a^2, b^2) + (a, bc) + (b, ac) + (c, ab) = 239^2$$

where (n, m) denotes the greatest common divisor of natural numbers n and m.

Solution: Suppose that $d > 1$ is a greatest common divisor of a, b, and c. Then d divides 239^2, and since 239 is a prime, $d \geq 239$. In this case $(a^2, b^2) \geq 239^2$, a contradiction. Thus, a, b, and c are relatively prime and $(a, bc) = (a, b) \cdot (a, c)$, $(b, ac) = (a, b) \cdot (b, c)$, $(c, ab) = (c, a) \cdot (c, b)$ and $(a^2, b^2) = (a, b)^2$. Now the equation has the following form:

$$\big((a, b) + (a, c)\big)\big((a, b) + (b, c)\big) = 239^2.$$

Since $(a, b) \geq 1, (a, c) \geq 1$, and $(b, c) \geq 1$, the only possibility is

$$(a, b) + (a, c) = 239 \text{ and } (a, b) + (b, c) = 239.$$

Since a, b, and c are relatively prime, we have $(a, c) = (b, c) = 1$ and therefore $(a, b) = 238$. Now note that $ab + c = 239^2$ and $(a, b) = 238$ leads to $a = b = 238$, otherwise $ab > 239^2$. Finally, $a = 238, b = 238, c = 477$ is the only solution.