Bilkent University
Department of Mathematics

Problem Of The Month

October 2005

Problem: Find all natural numbers n for which

$$\sqrt{1\underbrace{4\dots4}_{n\text{-times}}}$$

is an integer number.

Solution: $1\underbrace{4\dots4}_{n\text{-times}}$ is a perfect square for $n=2$ and $n=3$, $144=12^2$ and $1444=38^2$. Let us prove that for all values of $n \geq 4$, $1\underbrace{4\dots4}_{n\text{-times}}$ is not a perfect square. Suppose that $1\underbrace{4\dots4}_{n\text{-times}} = m^2$. Then $m=2k$ and direct substitution yields $k^2 = 36\underbrace{1\dots1}_{n-2\text{-times}} = 36\underbrace{1\dots1}_{n-4\text{-times}}00+11$. Modulo 4, right hand side is 3, but k^2 is 0 or 1 (indeed, if $k=4\ell+i, i=0,1,2,3$, then $k^2=4p+i^2=0,1,4,$ or 9 modulo 4). Contradiction.